AN APPLICATION OF MARKOVIAN CHAINS IN MANPOWER PLANNING

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Monterey, California

THESIS

AN APPLICATION OF MARKOVIAN CHAINS

IN MANPOWER PLANNING

by

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June 1977

Thesis Advisor: R. W. Butterworth

Approved for public release; distribution unlimited.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)

An Application of Markovian Chains in Manpower Planning

5. TYPE OF REPORT & PERIOD COVERED

Master's Thesis; June 1977

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)

Rashad Abdulmohymen Abu-Alsamh

8. CONTRACT OR GRANT NUMBER(s)

9. PERFORMING ORGANIZATION NAME AND ADDRESS

Naval Postgraduate School
Monterey, California 93940

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS

Naval Postgraduate School
Monterey, California 93940

12. REPORT DATE

June 1977

13. NUMBER OF PAGES

66

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

Naval Postgraduate School
Monterey, California 93940

15. SECURITY CLASS. (of this report)

Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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S/N 0102-014-6601 |

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

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by

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ABSTRACT

This thesis investigates the long run implications of short term policy making decisions. Manpower modeling techniques based on Markovian chains are applied to a small fast growing Navy organization. An interactive model is designed here to simulate policy changes and illustrate the interaction between flows, promotions and stocks, relative to a stable set of policy parameters over a period of time. Training and budget planning are addressed in some detail with a focus on the civilian population within the organization.

Because real data is not available, the numerical examples are based on hypothetical data. The examples are presented to give the reader a feeling for how the model simulates various policy alternatives.

TABLE OF CONTENTS

I. INTRODUCTION

This paper describes a mathematical approach to manpower planning for an emerging Navy. The situation is that of a small growing Navy with a comprehensive defense task. To expand and develop such a Navy to an appropriate size, a careful and detailed approach to manpower planning is necessary.

To make the best use of the very limited manpower resources available for the required rapid expansion, two policy decisions will be assumed:

1. All available domestic manpower resources would be utilized in military billets, mainly on board ships, while civilian billets in shore repair and logistic facilities would be filled by foreign imported expertise.

2. In the beginning, inputs to the enlisted personnel structure would be at several levels:

   a. At the lowest level where sailors make their way up via on the job training;

   b. At the middle level "PO_2" where sailors would come with some knowledge but lack experience.

The policy indicated above provides a solution for a rapidly growing Navy during the early years of its expansion program, but imposes serious long-term problems.

   a. The foreign labor force cannot be maintained indefinitely, but must be phased out smoothly according to some time table;

b. The comparatively large numbers of petty officers who join at the middle level would soon reach the top of the enlisted structure and will have to leave, making room for others to move up. If the only two exits for petty officers were to commissioned officers or outside the organization, the organization would lose needed experience.[1]

A solution for the two problems would be to direct the excessive enlisted personnel to the shore repair and logistic facilities as replacements for the foreign labor force according to the predetermined schedule.

It should be worth noting that some of the shore billets will require skills and backgrounds that would not be available within the enlisted force. Such billets will be filled from sources other than the enlisted force. One such source is the commissioned officers force. The comparatively large number of junior officers admitted to the organization in the early years and the anticipated accelerated promotion policy that is required to meet the expansion requirements are expected to force many officers out of the system prematurely. Those officers could be directed to the shore facilities.

---

[1]There are enough arguments that good CPO's do not usually make good ensigns due to the big difference in what is required of each; accordingly when a CPO is promoted to an ensign, the Navy is likely to lose the output of both.

II. MODEL OBJECTIVES

The model defined in this paper is designed to assist in management decisions with relation to manpower planning for policy making, training plans and budget planning.

A. POLICY MAKING

This area deals with decisions related to:

1. Recruiting

In this category questions might arise along the following lines:

a. How many men should be recruited every year and in what classes?

b. What would be the long run implications of different growth patterns if adopted?

c. Where is the present recruiting policy leading the force?

d. When should the recruiting policy be changed to help reach desired levels at certain points in time, and what should be the nature of the policy change?

2. Promotions

Questions may arise along the following lines:

a. What is the effect of maintaining the present promotion policy?

b. What happens to the grade mix if promotions were accelerated in the lower ranks and/or the higher ranks?

c. What happens to the grade mix if promotions were halted for some number of years either throughout the system or within certain ranks?

3. Retirements and Discharge

The focus in this area is on the systemic impact on the organization as a result of reducing the retiring age, thus allowing a larger number of people to leave the system from the comparatively lower ranks or vice versa.

B. TRAINING PLANS

This area deals with

1. Number of Trainees

a. What should be the capacity of the training facilities?

b. How would a change in the length of the training period affect the inventory at the training facilities, and the system.

2. Types of Training

This area deals with questions like:

a. How many people should be trained in what skills to meet a required skill mix by a certain time?

b. What is the impact of adding or eliminating one or more training curricula on the stock mix in the training facility and on the organization?

C. BUDGET PLANNING

The questions that may arise in this section are along the following lines:

1. How large is the fund required to cover the manpower inventory costs in a given period?

2. What are the financial impacts of any of the decisions made in the policy making or the training planning areas?

III. MODEL DESIGN

The Navy is comprised of the classes {recruits, sailors, petty officers, officers, civilians}. Movements from one class to another is permitted in some cases and prohibited in others. There are flows to and from the system and the outside on a continuous basis.

For simplicity, it will be assumed initially that the organization is partitioned into ten mutually exclusive states. Figure 1 is a diagrammatic representation of the system. The lines connecting any two blocks indicate personnel flows between those two blocks in the direction of the arrows. If we further assume that movements from one state to another or to the outside are all linear in nature, that is, in proportion to the number in the originating state each period, then the system can be modeled as a Markovian chain, represented by the P matrix in Figure 2.

The submatrix Q represents movements among the states. A letter q in the (i,j) position in the submatrix Q means a fraction q of the present stocks in the ith state will move to the jth state in each period of time. No letter q in the (i,j) position means that no flows are allowed between the states i and j. The (i,j) position is at the intersection of row i and column j. The submatrix R represents the outflows from the system. A letter r in the (i,j) position of the matrix R indicates a fraction r of the present stock

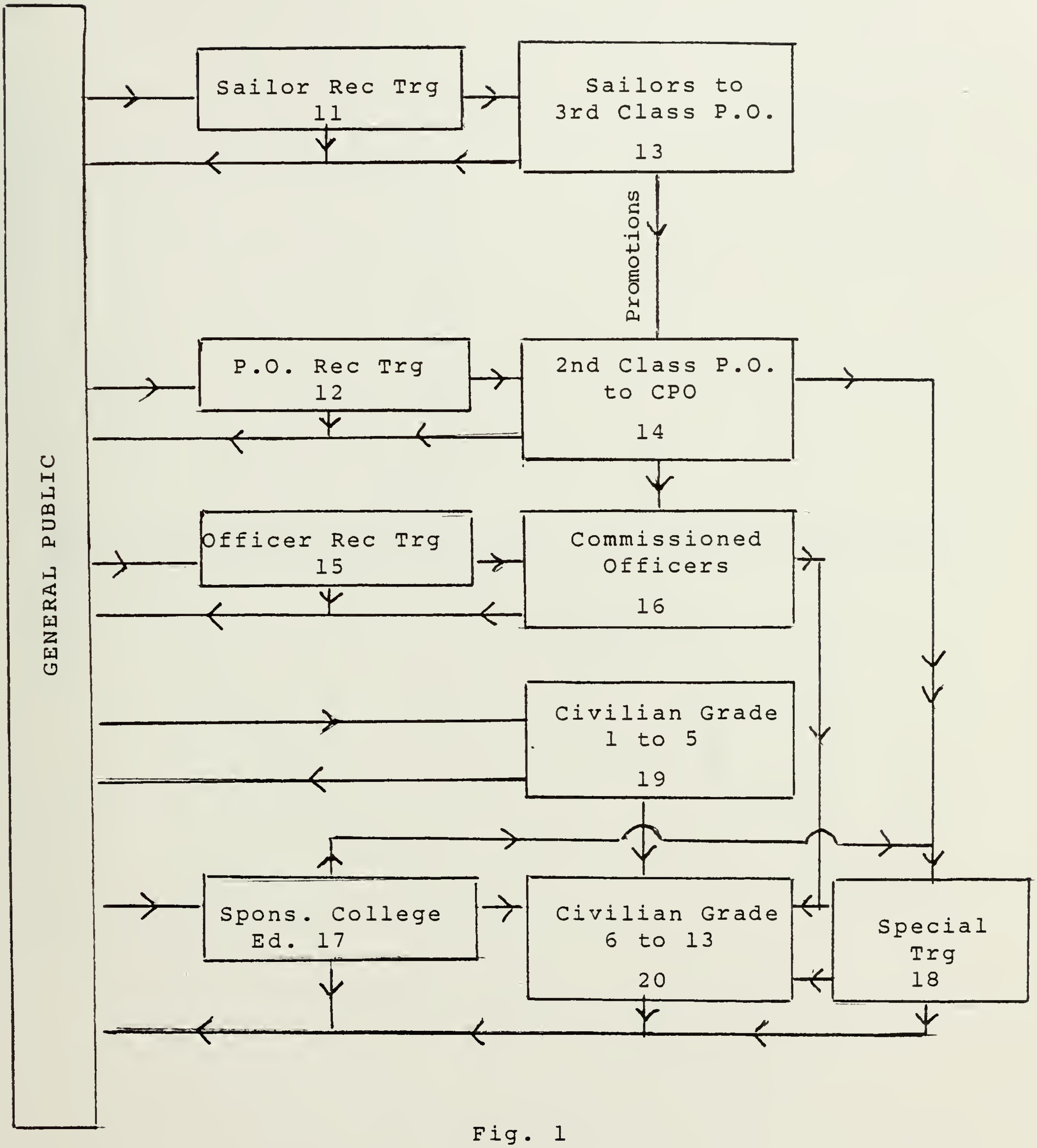
GENERAL PUBLIC
Sailor Rec Trg
11
Sailors to
3rd Class P.O.
13
Promotions
P.O. Rec Trg
12
2nd Class P.O.
to CPO
14
Officer Rec Trg
15
Commissioned
Officers
16
Civilian Grade
1 to 5
19
Spons. College
Ed. 17
Civilian Grade
6 to 13
20
Special
Trg
18

Fig. 1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	0	0	0	0	0	0	0	0	0										
2	0	1								0										
3	0		1							0										
4	0			1						0										
5	0				1					0										
6	0					1				0										
7	0						1			0										
8	0							1		0										
9	0								1	0										
10	0	0	0	0	0	0	0	0	0	1										
11	r										q		q							
12		r										q		q						
13			r										q	q						
14				r										q		q		q		
15					r										q	q				
16						r										q				q
17							r										q	q		q
18								r										q		q
19									r										q	q
20										r										q

Fig. 2

Note: 1. Refer to Fig. 1 for definition of states 11 thru 20.

2. States 1 thru 10 are holding states for the system outflows. State 1 receives outflows from state 11; state 2 receives outflows from state 12, and so on.

in the ith state will flow out to the jth state each period. It should be noted that the letters q and r in the P matrix indicate some positive fractions that sum to 1 on each row, but do not imply equal entries in the Q submatrix or in the R submatrix.

The inputs to the system from the outside are assumed here to be a vector $H = (h_{11}, h_{12}, \ldots . h_{20})$ that flows to the system each period.

Given the current stocks $S^{(0)} = (S_{11}, S_{12} \ldots . S_{20})$ and the matrix $P = \begin{bmatrix} I & O \\ R & Q \end{bmatrix}$ which quantifies the personnel management policy and the vector $H = (h_{11}, h_{12} \ldots . h_{20})$ which represents the direct input to the system each period, then the stocks after one period $S^{(1)}$ and after (t) periods $S^{(t)}$ as well as at the steady state $S^{(\infty)}$ can be calculated as follows [1]

$$\begin{aligned} S^{(1)} &= S^{(0)} \cdot Q + H \\ S^{(t)} &= S^{(0)} \cdot Q^{t} + H(Q^{t-1} + Q^{t-2} + \ldots + I) \\ S^{(\infty)} &= H \cdot (I-Q)^{-1} \end{aligned} \quad (1)$$

We can also calculate the number of people leaving the system during any given period.

Let $G^{(t)}$ be the number of people leaving the system during period (t).

$$\begin{aligned} G &= (g_1, g_2, \ldots . g_{10}) \\ G^{(1)} &= S^{(0)} R \\ G^{(t)} &= S^{(t-1)} \cdot R \\ G^{(\infty)} &= H \cdot (I-Q)^{-1} R \end{aligned} \quad (2)$$

As a timing convection, period t is used throughout this thesis to define the time interval (t-1,t], thus time t is the last instant in the period t, and time t-1 marks the beginning of period t, although it is not included in the period.

Ex. 1

Given the following information:

$H \quad = (100,\ 50,\ 0,\ 0,\ 20,\ 10,\ 10,\ 0,\ 10,\ 5)$

$S^{(0)} = (300,100,1000,400,60,200,20,40,20,30)$

P

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1																			
2		1																		
3			1																	
4				1																
5					1															
6						1														
7							1													
8								1												
9									1											
10										1										
11	.1										.1		.8							
12		.3										.2		.5						
13			.1										.4	.5						
14				.1										.6		.1		.2		
15					.1										.2	.7				
16						.15										.65				.2
17							.2										.5	.2		.1
18								.1										.1		.8
19									.2										.7	.1
20										.2										.8

Find $S^{(1)}$, $S^{(2)}$, $S^{(\infty)}$ and $G^{(1)}$, $G^{(2)}$, and $G^{(\infty)}$.

Table I

Solution: Applying the formulae in (1) and (2), the following results are obtained:

STATE	No.	H	$S^{(0)}$	$S^{(1)}$	$S^{(2)}$	$S^{(\infty)}$	$G^{(1)}$	$G^{(2)}$	$G^{(\infty)}$
Sailor Recruit Training	11	100.0	300.0	130.0	113.0	111.0	30.0	13.0	11.
P. O. Recruit Training	12	50.0	100.0	70.0	64.0	63.0	30.0	21.0	19.0
Lower rank sailors to PO_3	13	0.0	1000.0	640.0	360.0	148.0	100.0	64.0	15.0
PO_2 to CPO	14	0.0	400.0	790.0	829.0	263.0	40.0	79.0	26.0
Officer Recruit Training	15	20.0	60.0	32.0	26.0	25.0	6.0	3.0	3.0
Commissioned Officers	16	10.0	200.0	222.0	256.0	154.0	30.0	33.0	23.0
Sponsored College Ed.	17	10.0	20.0	20.0	20.0	20.0	4.0	4.0	4.0
Special Training	18	0.0	40.0	88.0	171.0	63.0	4.0	9.0	6.0
Civilian grades 1 to 5	19	10.0	120.0	94.0	76.0	33.0	24.0	19.0	7.0
Civilian grades 6 to 13	20	5.0	30.0	115.0	223.0	457.0	6.0	23.0	91.0

Note: Fractions are rounded to the nearest integer.

IV. AREAS OF APPLICATION

A. POLICY MAKING

1. Problem Formulation

This section is concerned with illustrating that by manipulating the elements of the submatrix Q and/or the vector H, one can simulate various policies and their probable effect on the stock levels. The focus here is on one part of the organization, namely the higher civilian grades, state no. (20).[2]

The main concern is to find answers to questions related to the projection of the manpower inventory into the future, given a certain policy, and a certain inventory level. Questions on the following lines:

a. What would the steady state inventory be?

b. What can be done to change the result in some desired direction?[3]

c. How fast, or slow are the stocks moving in the desired direction and what could be done to influence the rate of change?

---

[2]See example 1 for state definitions

[3]This paper addresses the mathematical side of the problem only. The reader is cautioned that changing the parameters of the model in real life situations would require careful feasibility analysis.

2. Assumptions

a. An ideal organization chart has been carefully developed in detail showing the ultimate size of the system under discussion and the required number of billets in each grade.

b. The promotion and attrition policies are stable over a fairly long period of time.

c. The annual inflows to the organization are constant over a fairly long period of time.

3. Mathematical Notation

Let $S_{20}^{(0)}$ be the current time stocks in the state 20 "civilian grades, 6 through 13" where $S_{20} = \{S_{201}, S_{202} \ldots \ldots S_{208}\}$ and

S_{201} = Grade 6

S_{202} = Grade 7

S_{203} = Grade 8

S_{204} = Grade 9

S_{205} = Grade 10

S_{206} = Grade 11

S_{207} = Grade 12

S_{208} = Grade 13

H_{20} = The annual inputs to the system where

$$H_{20} = \{h_{201}, h_{202} \ldots \ldots h_{208}\}$$

where h_{201} = the annual input to S_{201} "Grade 6"

h_{202} = the annual input to S_{202} "Grade 7"

$\vdots$

h_{208} = the annual input to S_{208} "Grade 13"

Let Q_{20} = 8x8 matrix that represents the promotion policy within the civilian higher grades 6 through 13 where q_{ij} = the fraction of the ith state that is assumed to move up to the jth state and q_{ij} will always equal zero except when j=i or j=i+1.

Let R_{20} = 8x8 diagonal matrix that represents the flows out of the system where r_i = the fraction of the stocks in the ith state that is allowed to flow out each period and

$$r_i + q_{i,i} + q_{i,i+1} = 1$$

Let the states {01, 02.....08} be holding states to receive the flows out of states 201, 202.....208 respectively. Now the system can be described by the diagram in Figure 3 where a line between two blocks indicates possibilities of personnel flows.

The matrix P_{20} in Figure 4 is the mathematical representation of the system; however it does not show the vector H_{20}. It should be kept in mind that H_{20} represents all inputs to the system i.e. direct input as well as the flows from the organization at large to the system under discussion, the state 20, civilian grades 6 through 13.

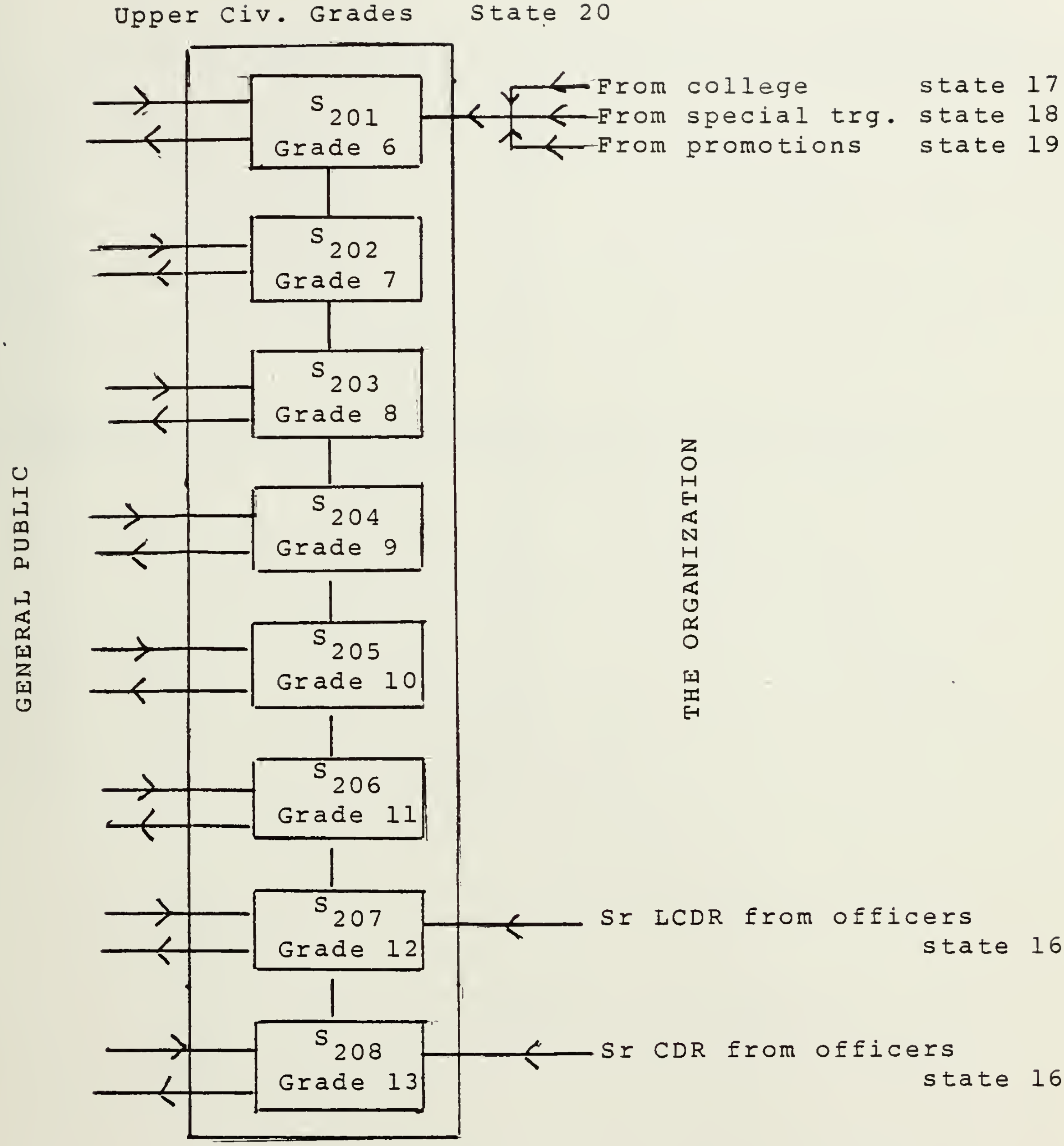
Upper Civ. Grades State 20
GENERAL PUBLIC
S_{201} Grade 6
S_{202} Grade 7
S_{203} Grade 8
S_{204} Grade 9
S_{205} Grade 10
S_{206} Grade 11
S_{207} Grade 12
S_{208} Grade 13
From college state 17
From special trg. state 18
From promotions state 19
THE ORGANIZATION
Sr LCDR from officers state 16
Sr CDR from officers state 16

Fig. 3

P_{20}

	01	02	03	04	05	06	07	08	201	202	203	204	205	206	207	208
01	l															
02		l														
03			l													
04				l												
05					l											
06						l										
07							l									
08								l								
201	r								q	q						
202		r								q	q					
203			r								q	q				
204				r								q	q			
205					r								q	q		
206						r								q	q	
207							r								q	q
208								r								q

Fig. 4

4. Model Application

The method is best exemplified by an interactive model where information is fed to a computer and varied by the operator to see the resulting patterns that follow a policy change. The following example demonstrates that.

Ex. 2

Given the following inputs

$$S_{20}^{(0)} = (2000,1500,800,800,600,300,100,100)$$

$$Q_{20}$$

$$\begin{array}{c} \\ 201 \\ 202 \\ 203 \\ 204 \\ 205 \\ 206 \\ 207 \\ 208 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 201 & 202 & 203 & 204 & 205 & 206 & 207 & 208 \end{array} \\
\begin{bmatrix}
.8 & .1 & & & & & & \\
 & .7 & .2 & & & & & \\
 & & .6 & .35 & & & & \\
 & & & .55 & .42 & & & \\
 & & & & .5 & .4 & & \\
 & & & & & .4 & .5 & \\
 & & & & & & .3 & .4 \\
 & & & & & & & .5
\end{bmatrix}
\end{array}$$

Calculate $S_{20}^{(1)}$ $S_{20}^{(2)}$ $S_{20}^{(3)}$ $S_{20}^{(4)}$ $S_{20}^{(5)}$ $S_{20}^{(\infty)}$ for the following cases:

	h_{201}	h_{202}	h_{203}	h_{204}	h_{205}	h_{206}	h_{207}	h_{208}
Case A	100	80	40	40	40	30	10	10
Case B	20	80	40	40	40	30	10	10
Case C	100	80	40	40	40	30	2	1

Table II

	Solution:	S_{201}	S_{202}	S_{203}	S_{204}	S_{205}	S_{206}	S_{207}	S_{208}
CASE A	$S^{(0)}$	2000	1500	800	800	600	300	100	100
	$S^{(1)}$	1700	1330	820	760	676	390	190	100
	$S^{(2)}$	1460	1181	798	745	697	456	262	136
	$S^{(3)}$	1268	1053	755	729	702	491	317	183
	$S^{(4)}$	1114	944	704	705	697	507	351	228
	$S^{(5)}$	992	852	651	674	685	512	369	264
	$S^{(\infty)}$	500	433	317	335	362	291	222	198
CASE B	$S^{(0)}$	2000	1500	800	800	600	300	100	100
	$S^{(1)}$	1620	1330	820	760	676	390	190	100
	$S^{(2)}$	1316	1173	798	745	697	456	262	136
	$S^{(3)}$	1073	1033	753	729	702	491	317	183
	$S^{(4)}$	878	910	699	705	697	507	351	228
	$S^{(5)}$	723	805	641	672	684	512	369	264
	$S^{(\infty)}$	100	300	250	283	318	262	201	181
CASE C	$S^{(0)}$	2000	1500	800	800	600	300	100	100
	$S^{(1)}$	1700	1330	820	760	676	390	182	91
	$S^{(2)}$	1460	1181	798	745	697	456	252	119
	$S^{(3)}$	1268	1053	755	729	702	491	306	161
	$S^{(4)}$	1114	944	704	705	697	507	339	204
	$S^{(5)}$	992	852	651	674	685	512	357	239
	$S^{(\infty)}$	500	433	317	335	362	291	211	171

Note: Fractions are rounded to nearest integer.

Discussion of example 2

Q_{20} is a hypothetical promotion matrix that allows slower promotions in the lower grades represented by states 201, 202 and 203 than the higher grades represented by states 204, 205, 206 and 207. For example, the first row of the matrix indicates that 80% of the stocks in state 201 remain there and only 10% are promoted to state 202 each period. Similarly, the 6th row indicates that only 40% of the stocks in state 206 stay there and 50% are promoted to state 207 each period. If such a policy were representative of the real situation one would expect Table II to be a representative of the stock inventory at the indicated time.

Case B in Table II shows what would be expected when inputs to the lowest grade suffer shortages, presumably as a result of shortages of the flows from other parts of the organization.[4]

Case C shows what happens when the input shortages take place at the upper grades, presumably as a result of shortage of flows from the other parts of the organization.

Case A acts as a reference line.

Comparing Case B to Case A line by line, one should notice that the reduction of input to the lowest grade, state 201, in Case B was immediately felt in that state only from the first year. Note that the second line, $S^{(1)}$, in

---

[4]The diagram in Figure 3 shows that large portion of the input to the lowest grade comes through states 17, 18 and 19. Should flows from such classes become short we should expect a pattern on the Case B lines.

Case A is identical to the second line, $S^{(1)}$, in Case B except for the element at the first column, S_{201}. After one more period the shortages are felt in the next higher state while the remaining six states 203 thru 208 show identical results under both cases. To verify this, compare the third line, $S^{(2)}$, in Case B to $S^{(2)}$ in Case A. It might also be useful to note that the effect of reducing input to the lowest grade upon the stocks in the other grades decreases in magnitude as it moves up the organizational hierarchy. Compare $S^{(\infty)}$ in Case B to $S^{(\infty)}$ in Case A; the lowest state 201 is reduced from 500 persons under Case A to only 100 under Case B; state 202 is reduced from 433 to 300 ..... and the highest state 208 is reduced from 198 to 181 only.

When the reduction of input occurs in the two highest grades only, states 207 and 208, as in Case C, obviously no changes take place in the preceding states 201 thru 206.

If for some reason the Q matrix is changed to reflect a policy of faster promotions in the lower grades than in the higher grades the projection of the inventory over some period of time will be much different.[5]

---

[5]While in real life a change of the promotion policy may have psychological impact on the human behavior which causes further changes in the elements of the mathematical model, such as the rate of attrition or therate of recruitment, it is assumed here that all other elements are held constant.

Q_{20} below is a reflection of such a policy with the assumption that attrition is held at the same level as in Q_{20}.

$$\bar{Q}_{20}$$

$$\begin{array}{cc}
 & \begin{array}{cccccccc} 201 & 202 & 203 & 204 & 205 & 206 & 207 & 208 \end{array} \\
\begin{array}{c} 201 \\ 202 \\ 203 \\ 204 \\ 205 \\ 206 \\ 207 \\ 208 \end{array} &
\begin{bmatrix}
.3 & .6 & & & & & & \\
 & .4 & .5 & & & & & \\
 & & .5 & .45 & & & & \\
 & & & .6 & .37 & & & \\
 & & & & .7 & .2 & & \\
 & & & & & .8 & .1 & \\
 & & & & & & .5 & .2 \\
 & & & & & & & .5
\end{bmatrix}
\end{array}$$

TABLE III

		S_{201}	S_{202}	S_{203}	S_{204}	S_{205}	S_{206}	S_{207}	S_{208}
CASE A	$S^{(0)}$	2000	1500	800	800	600	300	100	100
	$S^{(1)}$	700	1880	1190	880	756	390	90	80
	$S^{(2)}$	310	1252	1575	1104	895	493	94	68
	$S^{(3)}$	193	767	1454	1411	1075	604	106	63
	$S^{(4)}$	158	503	1150	1541	1314	728	124	63
	$S^{(5)}$	147	376	866	1482	1530	875	145	66
	$S^{(\infty)}$	143	276	356	501	751	901	200	100
CASE B	$S^{(0)}$	2000	1500	800	800	600	300	100	100
	$S^{(1)}$	620	1880	1190	880	756	390	90	80
	$S^{(2)}$	206	1204	1575	1104	895	493	94	68
	$S^{(3)}$	82	685	1430	1411	1075	604	106	63
	$S^{(4)}$	45	403	1097	1530	1314	728	124	63
	$S^{(5)}$	33	268	790	1452	1526	875	145	66
	$S^{(\infty)}$	29	162	242	372	592	742	169	87
CASE C	$S^{(0)}$	2000	1500	800	800	600	300	100	100
	$S^{(1)}$	700	1880	1190	880	756	390	82	71
	$S^{(2)}$	310	1252	1575	1104	895	493	82	53
	$S^{(3)}$	193	767	1454	1411	1075	604	92	44
	$S^{(4)}$	158	503	1150	1541	1314	728	109	41
	$S^{(5)}$	147	376	866	1482	1530	875	129	43
	$S^{(\infty)}$	143	276	356	501	751	901	184	76

Table IV

Policy	Case A	Case B	Case C
Q_{20}	2658	1890	2619
$\bar{Q}_{20}$	3228	2395	3188

Table V

		S_{201}	S_{202}	S_{203}	S_{204}	S_{205}	S_{206}	S_{207}	S_{208}
with	Case A	0.1881	0.163	0.1192	0.1261	0.136	0.1095	0.0836	0.0744
Q_{20}	Case B	0.05275	0.1582	0.1319	0.1494	0.1677	0.1382	0.1062	0.09554
	Case C	0.1909	0.1655	0.1209	0.128	0.138	0.1111	0.08046	0.06513
with	Case A	0.04426	0.08556	0.1103	0.1551	0.2326	0.2791	0.06201	0.031
$\bar{Q}_{20}$	Case B	0.01193	0.0676	0.101	0.1554	0.2473	0.3099	0.07034	0.03649
	Case C	0.04482	0.08665	0.1117	0.1571	0.2356	0.2826	0.05778	0.02374

Table III shows the solution of ex. 2 using $\bar{Q}_{20}$. A comparison between Tables II and III, line by line, shows that the fast promotions in the lower grades reduces the inventory of those grades considerably; however, contrary to one's intuition, no build up occurs in the higher grades. Table IV shows the total number of persons in the system at the steady state. It is worth noting that the total size is larger under the policy represented by $\bar{Q}_{20}$ than under the policy represented by Q_{20} in spite of keeping the attrition rates the same under the two policies.

Table V shows the percentage distribution of personnel over various grades in the steady state under each of the two policies and with each of the three cases.

The objective of example 2 and the above discussion is to illustrate that the size and composition of a personnel system is governed by flows in, flows out, and promotion policies jointly. To predict the long term impact of a change in a single area requires an understanding of the interaction among all the elements in the overall picture.

5. <u>Growth and Decay</u>

The discussion has so far addressed a constant size policy, which leads to a constant size organization, "steady state S^{∞}".

However, organizations in real life are either growing or decaying. An expansion of the previous discussion is necessary to show what would happen if the inflows were not constant but were varied at a constant rate.

a. Geometric Growth [2]

Let $H^{(0)}$ be the input vector in period 0 and let θ be the desired rate of growth, i.e. $\theta > 1$. Now the input vector at time t, $H^{(t)} = \theta^{t}H$, $t > 0$.

$$S^{(1)} = S^{(0)}Q + \theta H$$

$$S^{(2)} = S^{(0)}Q^{2} + \theta HQ + \theta^{2}H$$

$$S^{(t)} = S^{(0)}Q^{t} + H\sum_{j=0}^{t-1} \theta^{t-j}Q^{j} \qquad (3)$$

As t becomes large "i.e. approaches ∞" the expression converges to [2]

$$\underset{t\to\infty}{S^{(t)}} \approx \theta^{t}H\left(I - \frac{Q}{\theta}\right)^{-1} \qquad (4)$$

While the above formula provides insight into the future projections of the inventory level, given a geometric rate of growth θ, it further indicates that the equation exhibited under the constant size case was a special case of a geometric growth with $\theta = 1$.

b. Geometric Decay

If the organization were to contract in size, the rate of decay cannot be controlled by the value of θ beyond some limit. For example, if no flows were allowed into the system, making the organization die through natural attrition, this would imply that $\theta H = 0$, i.e. $\theta = 0$. With $\theta = 0$ the above formula would collapse. Thus the rate of decay must have an upper bound which cannot be exceeded and θ must have a lower bound beyond which it cannot go [2].

Ex. 3

Given the Q_{20} and $\bar{Q}_{20}$ matrices in example 2 show the stock levels at the end of each of the periods {1, 2, 3, 4, 5, 15, 25 and 26}.

Given $S^{(0)} = \{2000, 1500, 800, 600, 300, 100, 100\}$

$H^{(0)} = \{100, 80, 40, 40, 40, 30, 10, 10\}$

Rate of growth $\theta = 1.05$ (i.e. 5% growth)

Table VI

		eS	S_{201}	S_{202}	S_{203}	S_{204}	S_{205}	S_{206}	S_{207}	S_{208}
with policy matrix Ω	$S^{(0)}$	6200	2000	1500	800	800	600	300	100	100
	$S^{(1)}$	5984	1705	1334	822	762	678	392	191	101
	$S^{(2)}$	5787	1474	1193	804	751	703	461	264	137
	$S^{(3)}$	5599	1295	1075	767	741	713	500	321	186
	$S^{(4)}$	5416	1158	979	724	725	716	522	359	234
	$S^{(5)}$	5243	1054	903	681	703	714	534	381	273
	$S^{(15)}$	4883	929	802	578	611	661	535	406	362
	$S^{(25)}$	7128	1428	1225	864	893	945	750	551	471
	$S^{(26)}$	7466	1498	1285	905	936	990	785	576	491
with policy matrix Ω'	$S^{(0)}$	6200	2000	1500	800	800	600	300	100	100
	$S^{(1)}$	5984	705	1884	1192	882	758	392	91	81
	$S^{(2)}$	5842	322	1265	1582	1110	901	498	95	69
	$S^{(3)}$	5773	212	792	1470	1424	1088	613	109	65
	$S^{(4)}$	5746	185	541	1179	1564	1337	745	128	67
	$S^{(5)}$	5735	183	430	911	1520	1566	901	151	72
	$S^{(15)}$	5906	291	537	648	864	1338	1674	372	181
	$S^{(25)}$	8308	474	875	1054	1370	1863	2001	440	231
	$S^{(26)}$	8690	498	919	1107	1439	1953	2080	456	239

TABLE VII

The percentage distribution over grades.

		s_{201}	s_{202}	s_{203}	s_{204}	s_{205}	s_{206}	s_{207}	s_{208}
	$s^{(0)}$	0.323	0.242	0.129	0.129	0.097	0.048	0.016	0.016
	$s^{(1)}$	0.285	0.223	0.137	0.127	0.113	0.065	0.032	0.017
	$s^{(2)}$	0.255	0.206	0.139	0.13	0.121	0.08	0.046	0.024
	$s^{(3)}$	0.231	0.192	0.137	0.132	0.127	0.089	0.057	0.033
Q	$s^{(4)}$	0.214	0.181	0.134	0.134	0.132	0.096	0.066	0.043
	$s^{(5)}$	0.201	0.172	0.13	0.134	0.136	0.102	0.073	0.052
	$s^{(15)}$	0.19	0.164	0.118	0.125	0.135	0.109	0.083	0.074
	$s^{(25)}$	0.2	0.172	0.121	0.125	0.133	0.105	0.077	0.066
	$s^{(26)}$	0.201	0.172	0.121	0.125	0.133	0.105	0.077	0.066
	$s^{(0)}$	0.323	0.242	0.129	0.129	0.097	0.048	0.016	0.016
	$s^{(1)}$	0.118	0.315	0.199	0.147	0.127	0.065	0.015	0.013
	$s^{(2)}$	0.055	0.216	0.271	0.19	0.154	0.085	0.016	0.012
	$s^{(3)}$	0.037	0.137	0.255	0.247	0.188	0.106	0.019	0.011
$\bar{Q}$	$s^{(4)}$	0.032	0.094	0.205	0.272	0.233	0.13	0.022	0.012
	$s^{(5)}$	0.032	0.075	0.159	0.265	0.273	0.157	0.026	0.013
	$s^{(15)}$	0.049	0.091	0.11	0.146	0.227	0.283	0.063	0.031
	$s^{(25)}$	0.057	0.105	0.127	0.165	0.224	0.241	0.053	0.028
	$s^{(26)}$	0.057	0.106	0.127	0.166	0.225	0.239	0.052	0.027

Discussion of Example 3

Table IV shows that stocks started decaying in spite of the .05 desired rate of growth. This is because the initial flows in vector H are much more less than the flows out; thus the system showed decay symptoms. To explain this further, consider S_{201} under the Q policy between periods zero and one. S_{201} starts with 2000 at time zero and reaches 1705 at time one. Referring to Q_2 in example 2 one finds that 0.8 stay, 0.1 get promoted, 0.1 move out, and 100 persons flow in each period. This means 0.2 of 2000 which is 400 persons move out and only 100 persons move in, in that period. It is clear that increasing the 100 to balance the 400 requires many periods, if the growth rate is 5% only. Now consider the same state between periods 25 and 26. S_{201} reached 1428 at time 25; the flows out are $0.2 \times 1428 = 285.6$, and the flows in are $(1.05)^{26} \times 100 = 355.6$, i.e. a net increase of 70 persons are added to the inventory of time 25 which makes 1498 persons at the next period, time 26. After the growth rate catches up with the decay, then the stocks start growing at the constant rate governed by θ. It can easily be verified that the stocks at time 26 are 1.05 times the stocks at time 25.

It is further noted that, while the stocks keep growing at the constant rate they tend to maintain the same composition, i.e.

$$\frac{s_i^{(t)}}{eS^{(t)}} \text{ becomes stable as t grows}$$

indefinitely, where $eS^{(t)} = \sum_{i=1}^{N} s_i^{(t)}$. However for this

stable condition to occur, t must be large enough. In example 3, Table VII indicates that the system reached that condition under the policy represented by Q_{20} at time 25. It can readily be seen that $S^{(25)}$ has the same composition as $S^{(26)}$. An interested reader may calculate $S^{(t)}$ for any t larger than 26 using equation (4) to verify that the composition does not change.

It is worth noting that Table VII shows under the promotion policy represented by $\bar{Q}_{20}$ that the system becomes very close to the stable composition at time 25 but needs some more time to stabilize.

B. TRAINING PLANS

1. Problem Formulation

The focus here is on the training aspect of the system. The model will monitor the quantities and qualities[6] of the training and will help predict the impact of changing the training policy on the training facilities as well as on the entire system. This discussion first addresses the quantity issue to answer questions about the capacity of the training facility and its relation to the length of training.

2. Assumptions

a. The state 18 "special training" is the main training facility in the system.

b. Individuals are trained there over a range of skills. A range of five skills is considered here for illustration purposes.

c. Different skills require different lengths of training time; some require a one year program and some require a two year program.

d. Trainees come to the training facility from two sources namely, CPOs from the petty officer class, state 14, and college graduates from the sponsored college training class, state 17.

---

[6] Quantities here denote aggregate number of individuals in the training facilities while qualities denote the number of different technical curricula to be taught in the training facilities.

e. Movements through and out of the training system follow fixed rules that are stable over a fairly long period of time.

3. <u>Mathematical Notation</u>

Let $E^{(t)}$ be the number of chief petty officers that flows to the training facility at time t

$U^{(t)}$ be the number of college graduates who flows to the training facility at time t

$W^{(t)} = E^{(t)} + U^{(t)}$ = total inflows at time t

$\alpha W^{(t)}$ = the fraction of the total input that flows to the one year program skills and

$(1-\alpha)W^{(t)}$ the fraction that flows to the two year program skills.

Let the training system S_{18} be partitioned as follows:

S_{181} = Individuals enrolled in the one year program

S_{182} = Individuals in the junior year of the two year program

S_{183} = Individuals in the senior year of the two year program.

Let states g_{01}, g_{02} and g_{03} be holding states receiving the flows out of the system as follows:

g_{01} = Individuals dropped out of training

g_{02} = Individuals graduating from the one year program

g_{03} = Individuals graduating from the two year program

Let H be a row vector of flows into the system

$$H = \{\alpha W, (1-\alpha)W, 0\}^{7}$$

Now the training system can be represented as the diagram in Figure 5 or equally the P_{18} matrix in Figure 6.

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[7] The third element of H is zero, indicates no flows are allowed from outside the training system directly to state 183, the senior year of the two year program.

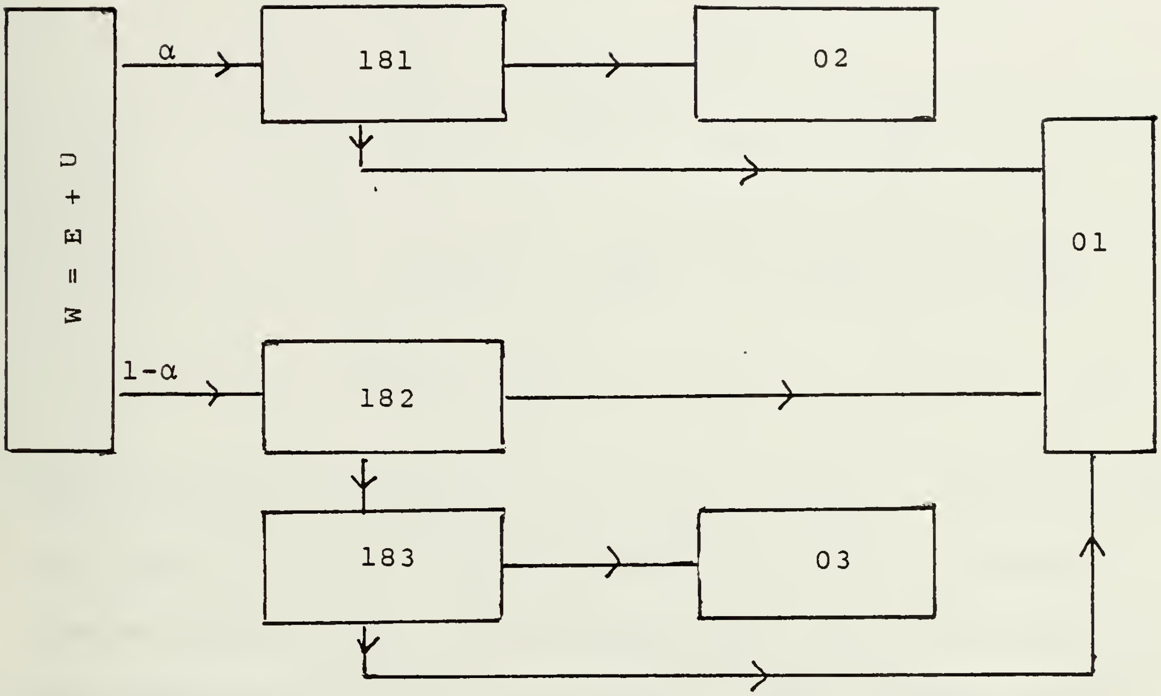


Fig. 5

	01	02	03	181	182	183
01	1					
02		1				
03			1			
181	r	r		q		
182	r				q	q
183	r		r			q

Fig. 6

Then

$$S_{18}^{(1)} = S^{(0)}\underset{\sim}{Q} + H^{(1)}$$

$$S^{(2)} = S^{0}\ \underset{\sim}{Q}^{2} + H^{(1)}\underset{\sim}{Q} + H^{(2)}$$

$$S^{(3)} = S^{0}\ \underset{\sim}{Q}^{3} + H^{1}\underset{\sim}{Q}^{2} + H^{(2)}\underset{\sim}{Q} + H^{(2)}$$

$$S^{(4)} = S^{0}\ \underset{\sim}{Q}^{4} + H_{18}^{(1)}\underset{\sim}{Q}^{3} + H_{18}^{(2)}\underset{\sim}{Q}^{2} + H_{18}^{(3)}\underset{\sim}{Q} + H_{18}^{(4)}$$

$$\vdots$$

$$S^{(t)} = S^{0}\ \underset{\sim}{Q}^{t} + H_{18}^{(1)}\underset{\sim}{Q}^{t-1} + H_{18}^{(2)}\underset{\sim}{Q}^{t-2} + H_{18}^{(3)}\underset{\sim}{Q}^{t-3} + \ldots H_{18}^{(t)}$$

$$= S^{0}\ \underset{\sim}{Q}^{t} + \sum_{i=1}^{t} H^{(i)}\underset{\sim}{Q}^{t-i} \qquad (5)$$

Now the variable elements of H are E and U. To obtain them, the state 14, petty officers, and the state 17, college graduates, have to be broken into their respective components. Under equilibrium conditions, those states become constant in size, hence E, U, and W become constant and the equation becomes

$$S_{18}^{(\infty)} = H_{18}(I-\underset{\sim}{Q}_{18})^{-1}$$

$$G^{(t)} = S_{18}^{(t-1)} \cdot R_{18}$$

$$\text{and} \quad G^{(\infty)} = H_{18}(I-\underset{\sim}{Q}_{18})^{-1} \cdot R_{18}$$

$$\text{where} \quad G = (g_{01}, g_{02}, g_{03})$$

4. Model Application

The application here is also on an interactive basis. The model can be used to iterate a current situation for a number of years and display the stocks and the outputs relevant to each future year, thus the decision maker can see the peaks and valleys in either the stocks or the output and

can simulate a corrective action and see the probable result. Another useful application is the simulation of changing the length of training to see the impact on the projected output of the system.

Ex. 4

Table VIII below shows the values of E and U as projected from a computer printout for the periods indicated.[8] Given the stock vector $S^{(0)}$ shown below, and the system's matrix P_{18} in Figure 7, and knowing that $\alpha = 0.3$, calculate:

a. The stocks in the system at the end of each of the periods indicated.

b. The graduates from each program.

Table VIII

Periods(i) / Elements	1	2	3	4	5	6	∞
E	60	62	70	90	115	120	130
U	0	0	10	25	25	15	20
W	60	62	80	115	140	135	150

$S^{(0)} = 50, 60, 70$

	01	02	03	S181	S182	S183
01	1					
02		1				
03			1			
S181	.05	.85		.1		
S182	.2				.2	.6
S183	.05		.9			.05

Fig. 7

[8] Hypothetical data.

Solution: $H^{(1)}$ = (60x.3, 60x.7, 0) = (18 42 0)

$H^{(2)}$ = (62x.3, 62x.7, 0) = (18.6 43.4 0)

$H^{(3)}$ = (80x.3, 80x.7, 0) = (24 56 0)

$H^{(4)}$ = (115x.3, 115x.7, 0) = (34.5 80.5 0)

$H^{(5)}$ = (140x.3, 140x.7, 0) = (42 98 0)

$H^{(6)}$ = (135x.3, 135x.7, 0) = (40.5 94.5 0)

$H^{(\infty)}$ = (150x.3, 150x.7, 0) = (45 85 0)

Table IX

Periods / States		1	2	3	4	5	6	∞
one year prog	S_{181}	23	20.9	26.09	37.11	45.71	45.07	49.75
junior year	S_{182}	54	54.2	66.84	93.87	116.77	117.85	106.25
senior year	S_{183}	39.2	34.38	34.24	41.82	58.41	72.98	67.07
dropouts	g_{01}	18	13.91	13.6	16.38	22.72	28.56	27.1
one year grad	g_{02}	42.5	19.55	17.77	22.18	31.54	38.85	42.46
two year grad	g_{03}	63	35.28	30.94	30.82	37.64	52.57	60.36

Decisions that affect the elements of the Q_{18} matrix or the $H_{18}^{(i)}$ vector would have results along the lines shown in examples 2 and 3, illustrated earlier; however varying the time length of training would lead to situations the planner might want to consider.

Ex. 5

Suppose in the previous example the two year program was extended one more period, i.e. those who succeed in the second year of training do not graduate but enter a third year of advanced training. At the end of the third year .07 would repeat the training, .03 would dropout and .9 graduate. What would the new flows and stocks look like over the same period of time, given the same data in ex. 4, after adding a zero

element to be the fourth element in each of the row vectors H and $S^{(0)}$?

Solution:

The new P_{18} matrix would be as follows

$$
\begin{array}{c|ccc|cccc}
 & 01 & 02 & 03 & 181 & 182 & 183 & 184 \\
01 & 1 & & & & & & \\
02 & & 1 & & & & & \\
03 & & & 1 & & & & \\
\hline
181 & .05 & .85 & & .1 & 0 & 0 & 0 \\
182 & .2 & & & 0 & .2 & .6 & 0 \\
183 & .03 & & .9 & 0 & 0 & 0 & .07
\end{array}
$$

Fig. 8

The states would be defined as follows

S_{183} individuals in the second year of the three year program

S_{184} individuals in the final year of the three year program

g_{03} individuals who graduate from the three year program.

All other states keep the same definitions as ex. 5.

Table X

		1	2	3	4	5	6	∞
one year prog	S_{181}	23	20.9	26.09	37.11	45.71	45.07	50
1st year of 3 year prog	S_{182}	54	54.2	66.84	94.23	116.85	117.87	106
2nd year of 3 year prog	S_{183}	39.5	34.38	34.24	41.82	58.63	73.04	67.1
3rd year of 3 year prog	S_{184}	63	72.36	68.53	35.61	40.13	55.58	64.9
dropouts	g_{01}	18	15.82	15.77	18.44	23.86	29.79	29.0
one year grad	g_{02}	42.5	19.55	17.77	22.18	31.54	38.85	42.5
3 year grad	g_{03}	0	56.7	65.12	61.68	32.05	36.12	58.41

Table XI

Comparison between stock
under the two different policies

	1	2	3	4	5	6	∞
Policy I	116.2	109.48	127.17	177.8	220.89	235.9	223.07
Policy II	179.5	181.36	195.7	208.77	261.32	291.56	288.0

Table XII

Comparison between training output[9]
under the two different policies

	1	2	3	4	5	6	∞
Policy I	105.5	54.83	48.71	53.00	69.18	91.38	102.82
Policy II	42.5	76.25	82.89	83.86	63.59	74.97	100.91

Tables IX, X, XI and XII show that under almost the same advancement policy and with the same inputs of trainees, the training facility in the previous examples would turn out much different each year and require a different size capacity.

This type of information would help the planner better manage the interface between the training program and the manpower requirements of the organization as the output of the training system comprises an important portion of the input to the organization. This leads to the following discussion.

---

[9] Training output = $g_{02} + g_{03}$.

5. <u>The Partition by Skills</u>:

The number of graduates from each program might not be sufficient. The planner may need to know the levels of stocks and flows by skill. To get these details, he will have to expand the input row vector H into a matrix $\underline{H}$, an M x N matrix, where M is the number of skills in the program that contains the largest number of skills and N is the number of programs in the training system.

Let $\underline{h}_j$ be the jth column in the matrix $\underline{H}$. This is the breakdown of the jth element in the row vector H by skills. Whenever the jth element of the row vector H is zero, i.e. no input from outside to the state j, the column $\underline{h}_j$ will be all zeros,

$$\text{e.g.} \quad H = (h_1 \ , \ h_2 \ , \ 0)$$

$$\underline{H} = \begin{bmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ h_{31} & h_{32} & 0 \\ h_{41} & 0 & 0 \\ h_{51} & 0 & 0 \end{bmatrix}.$$

and

$$e\underline{h}_1 = \sum_{i=1}^{M} h_{i1} \equiv h_1, \quad e\underline{h}_2 = \sum_{i=1}^{M} h_{i2} \equiv h_2, \quad e\underline{h}_j = \sum_{i=1}^{M} h_{ij} \equiv h_j$$

where e is an M row vector of ones

$$e = (1, 1, \ldots .1)$$

The $\underline{H}$ matrix is the partition of all joining individuals by program and by skill, so the h_{ij} element is the number of

people joining the ith skill in the jth program. The distribution of h_j over the elements of $\underline{h}_j$ is according to an allocation policy represented by an M column vector a_j.

$$a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{Mj} \end{pmatrix} \quad \text{where} \quad 0 \leq a_{ij} \leq 1 \text{ and } ea_j = 1$$

the number of individuals allowed in skill i of program

$j = h_j \; a_{ij}$.

Let each program be represented by a state, then, the row vector S can be expanded to be the matrix $\underline{S}$ with the same size as $\underline{H}$. Now substituting $\underline{S}$ for S and $\underline{H}$ for H in the equation, the results would be in a matrix form showing the numbers of individuals in each program of each skill. The following example will show an expansion of example 4 along the skills dimension.

Ex. 6

Given the input vectors $H^{(1)}$ in ex. 5 and the fact that the annual input to the one year program was usually distributed over the three skills covered by that program as follows

$$a_1 = \begin{pmatrix} a_{11} = .3 \\ a_{21} = .2 \\ a_{31} = .5 \\ a_{41} = 0 \end{pmatrix}$$

and the inputs to the two year program were distributed over the four skills covered by it as follows

$$a_2 = \begin{pmatrix} a_{12} = .2 \\ a_{22} = .3 \\ a_{32} = .4 \\ a_{42} = .1 \end{pmatrix}$$

and the current stocks in the training facility were represented by the matrix $\underline{S}^{(0)}$

S_{181}	S_{182}	S_{183}
15	12	14
10	18	21
25	24	28
0	6	7

The policy matrix P_{18} was still in effect. Find the expanded solutions over the first period.

Solution:

M is the largest number of skills in any one program.

$M = 4, N = 3$

$\underline{H} = 4x3$

$H^{(1)} = (h_1 = 18, h_2 = 42, h_3 = 0)$

$$\begin{array}{lllll} & & & & \underline{h}_1 \\ \underline{h}_1 & = & h_{11} & = 18 \times .3 = & 5.4 \\ & & h_{21} & = 18 \times .2 = & 3.6 \\ & & h_{31} & = 18 \times .5 = & 9.0 \\ & & h_{41} & = 18 \times 0 = & 0 \end{array}$$

$$\begin{array}{lllll} & & & & \underline{h}_2 \\ \underline{h}_2 & = & h_{12} & = 42 \times .2 = & 8.4 \\ & & h_{22} & = 42 \times .3 = & 12.6 \\ & & h_{32} & = 42 \times .4 = & 16.8 \\ & & h_{42} & = 42 \times .1 = & 4.2 \end{array}$$

$$\underline{h}_3 = 0 = \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}$$

$$\underline{H}^{(1)} = \begin{bmatrix} 5.4 & 8.4 & 0 \\ 3.6 & 12.6 & 0 \\ 9.0 & 16.8 & 0 \\ 0.0 & 4.2 & 0 \end{bmatrix}$$

$$\underline{S}^{(1)} = \underline{S}^{(0)}Q + \underline{H}$$

$$\underline{S}^{0}Q = \begin{array}{c} \begin{array}{ccc} 181 & 182 & 183 \end{array} \\ \begin{bmatrix} 15 & 12 & 14 \\ 10 & 18 & 21 \\ 25 & 24 & 28 \\ 0 & 6 & 7 \end{bmatrix} \end{array} \begin{bmatrix} .1 & 0 & 0 \\ 0 & .2 & .6 \\ 0 & 0 & .05 \end{bmatrix}$$

$$\underline{S}^{0}Q = \begin{bmatrix} 1.5 & 2.4 & 7.9 \\ 1.0 & 3.6 & 11.85 \\ 2.5 & 4.8 & 15.8 \\ 0 & 1.2 & 3.95 \end{bmatrix}$$

$$\underline{S}^{(1)} = \begin{bmatrix} 6.9 & 10.8 & 7.9 \\ 4.6 & 16.2 & 11.85 \\ 11.5 & 21.6 & 15.8 \\ 0 & 5.4 & 3.95 \end{bmatrix}$$

Note that $e\underline{S}^{(1)} = S^{(1)}$

$$e\underline{S}^{(1)} = (23.0 \quad 54.0 \quad 39.50)$$

The above vector $e\underline{S}^{(1)}$ can be easily compared to $S^{(1)}$ in example 4 above to show identical results; however the matrix $\underline{S}^{(1)}$ reveals more details than the vector $S^{(1)}$. Solutions for all of the other periods can be found similarly.

The user must be cautioned that all skills in a particular state share the same q and r which implies an assumption of no segregation among the skills in one grade with respect of advancement and attrition from that grade. Whenever such an assumption is not realistic, more states have to be added to the Q matrix.

6. Interface with Organization

Graduates from the training facilities, state 18, join the organization through the civilian grades, state 20.

They all join in grade 6, state 201, in the appropriate skills. This requires that the input row vector

$$H_{20} = \{h_{201}, h_{202}, \ldots h_{208}\}$$

be expanded to the matrix $\underline{H}_{20}$ of size M x N where M is the largest number of skills in one grade and N = the number of grades. Likewise S_{18} is expanded to be the matrix $\underline{S}_{18}$ of size M x N.

This arrangement will have no effect on the size of Q_{18} and the model for this particular state will be

$$\underline{S}_{18}^{(t)} = \underline{S}_{18}^{(t-1)} \; Q_{18} + \underline{H}_{18}$$

In the steady state

$$\underline{S}_{18}^{\infty} = \underline{H}_{18}(I-Q_{18})^{-1} \qquad (6)$$

Likewise the flows out of the system would be in matrix form, where $\underline{G}_{18}$ is an M x N matrix,

$$\underline{G}_{18}^{\infty} = \underline{H}_{18}(I-Q_{18})^{-1} \; R_{18} \qquad (7)$$

Thus inflow, outflows, and stocks can be identified and carried forward from one period to another in matrix form. This shows numbers of individuals in each skill in each grade.

While the above arrangement gives useful details, it has two implicit additional assumptions.

a. All skills in any grade are treated the same with respect to promotion and attrition, i.e. regardless

of the type of skill a person has, he is entitled to the same chance for promotion and attrition.

b. The above assumption leads to another one that says "every person has a chance to reach the top of the organization, governed only by his grade, regardless of his skill".

In reality, the assumptions above may very likely be untrue. The model can be adapted to accommodate the real situation by increasing the number of states N. Thus dividing S_i into two states S_{ia} and S_{ib} where the skills in one grade i are split in two groups a and b, so that each group might have a chance different than the other group for promotion to next grade (i+1). This will result in more realistic projection at the price of increasing the size of Q_{18}, R_{18}, $\underline{H}_{18}$ and $\underline{S}_{18}$. The same concept may be used to allow for blocking certain skills from moving up the grade hierarchy. This could be done by designating the highest allowable grade for some skills a blocking grade and dividing it into two states S_{ia} and S_{ib} where all skills move from S_{i-1} to S_{ia} or S_{ib} with equal chance, then those in S_{ia} can continue to move up to $S_{(i+1)}$ while those in S_{ib} are blocked.

C. BUDGET PLANNING

The objective of this section is to find ways for calculating the budget required for various inventory levels, and the effect of policy changes on the budget requirements.

The discussion will show how models can be used for the calculations and will address the financial impact of the policy changes.

1. Problem Formulation

For the civilian population within the Navy, budget is most affected by:

a. The number of people in each grade

b. The personnel time in service

c. The different types of skills within each grade

The system needs to be quantified along these three dimensions. In the first part of this section, the vector S = $(S_1, S_2, \ldots S_N)$ was a representation of the system along one dimension, the pay grades. In the second part, the vector S was expanded into a matrix $\underline{S} = M \times N$ to represent the system along the N grades and the M skills. In this part, another dimension is introduced to represent the system along the length of service using the three dimensional array $L = M \times N \times K$ where K = the highest level of salary allowed in any grade,[10]

and $$L^{(t)} = L_1^{(t)}, L_2^{(t)}, L_3^{(t)} \ldots\ldots L_K^{(t)}$$

where $\ell_{(i,j,k)}^{(t)}$ = number of persons in skill i, grade j and pay level k, at the conclusion of period t.

---

[10] Salary levels are arranged such that a person's salary moves up one level each period to a maximum of K levels.

2. Mathematical Notation

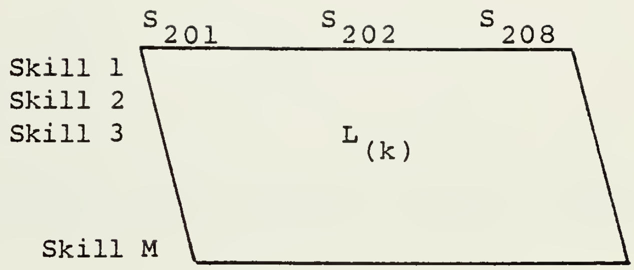
Let $L^{(t)}_{(1)} = H_{20}$

$L^{(t)}_{(2)} = H_{20}Q_{20}$

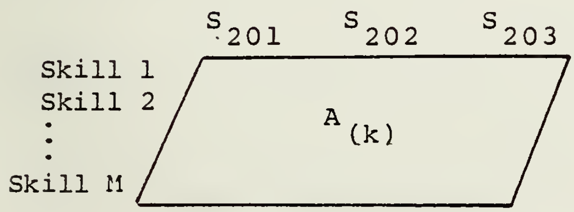
$L^{(t)}_{(3)} = H_{20}Q^{2}_{20}$

$\vdots$

$L^{(t)}_{K} = S^{(t)}_{20} - \sum_{k=1}^{K-1} L^{(t)}_{k}$



Let $A_{(k)}$ be an N x M matrix where $a_{(i,j,k)}$ = the allowance payable to individuals in skill i, grade j and a salary level k, k = [1,2,...,K].



Now the annual allowances payable to all personnel over the t^{th} period equals $B^{(t)}_{a}$ where

$$B^{(t)}_{a} = \sum_{k=1}^{K} \sum_{j=1}^{N} \sum_{i=1}^{M} \ell^{(t)}_{(i,j,k)} \, a_{(i,j,k)} \qquad (8)$$

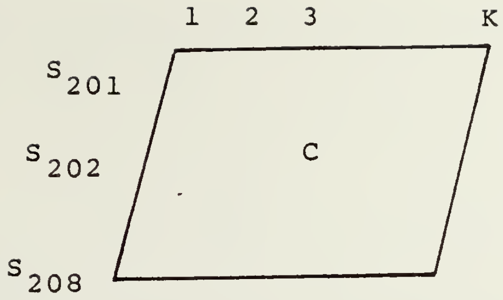
The salaries vary according to grades only, regardless of the type of skills. Let C be an N x K matrix where

N = 1,2,...,N grades

K = 1,2,...,K pay levels and

c_{ij} = the annual salary that is payable to grade i of pay level j where

j = 1,2,...,K.



The salary payable to all personnel over the t^{th} period equals $B_c^{(t)}$ where

$$B_c^{(t)} = \sum_{k=1}^{K} \sum_{j=1}^{N} \left[\sum_{i=1}^{M} L_{(i,j,k)}^{(t)} \right] C_{jk} \qquad (9)$$

The budget estimate is the sum of equations (8) and (9). It is denoted by $B^{(t)}$

$$\text{where} \quad B^{(t)} = B_a^{(t)} + B_c^{(t)} \qquad (10)$$

3. Model Application

The model has been modified in the previous section to yield information about the budget. The main question raised here is the budget size required, given certain policy, salaries and allowances table.

Matrices A and C, described earlier, would capture the financial data and would vary the model output according to the variations in the financial data input. The matrix L_k would capture the manpower policy changes through its H and/or Q components.

Once all variables are entered in the formulas, then the budget estimation becomes a matter of programming.

Ex. 8 below illustrates an eight grade system with six skills and five levels of salary.

Ex. 7

An organization starting with

1) The inventory stocks $\underline{S}^{(0)}$ as follows

$$\begin{array}{l c}
 & \begin{array}{cccccccc} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \end{array} \\
\begin{array}{l} \text{Skill 1} \\ \text{Skill 2} \\ \text{Skill 3} \\ \text{Skill 4} \\ \text{Skill 5} \\ \text{Skill 6} \end{array} &
\left[\begin{array}{cccccccc}
40 & 30 & 20 & 25 & 20 & 18 & 15 & 10 \\
30 & 25 & 15 & 20 & 20 & 16 & 20 & 8 \\
20 & 25 & 10 & 15 & 18 & 16 & 18 & 8 \\
20 & 10 & 10 & 15 & 16 & 12 & 12 & 6 \\
0 & 0 & 20 & 18 & 12 & 10 & 6 & 4 \\
0 & 0 & 0 & 12 & 8 & 8 & 4 & 2
\end{array}\right]
\end{array}$$

2) The annual flows $\underline{H}$ =

$$\begin{array}{l c}
 & \begin{array}{cccccccc} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 \end{array} \\
\begin{array}{l} \text{Skill 1} \\ \text{Skill 2} \\ \text{Skill 3} \\ \text{Skill 4} \\ \text{Skill 5} \\ \text{Skill 6} \end{array} &
\left[\begin{array}{cccccccc}
30 & 28 & 18 & 20 & 18 & 16 & 12 & 8 \\
28 & 24 & 16 & 18 & 10 & 12 & 10 & 6 \\
18 & 22 & 12 & 16 & 14 & 10 & 8 & 8 \\
16 & 12 & 10 & 10 & 8 & 8 & 6 & 8 \\
0 & 0 & 8 & 8 & 6 & 6 & 3 & 4 \\
0 & 0 & 0 & 6 & 4 & 6 & 7 & 4
\end{array}\right]
\end{array}$$

3. The Q matrix =

$$\begin{array}{l c}
 & \begin{array}{cccccccc} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \end{array} \\
\begin{array}{l} S1 \\ S2 \\ S3 \\ S4 \\ S5 \\ S6 \\ S7 \\ S8 \end{array} &
\left[\begin{array}{cccccccc}
.75 & .2 & & & & & & \\
 & .75 & .2 & & & & & \\
 & & .75 & .2 & & & & \\
 & & & .75 & .2 & & & \\
 & & & & .7 & .1 & & \\
 & & & & & .7 & .1 & \\
 & & & & & & .5 & .05 \\
 & & & & & & & .4
\end{array}\right]
\end{array}$$

4. The basic salary table (in thousands of dollars) table 8,

Table XIII

	Level 1	Level 2	Level 3	Level 4	Level 5
S_1 = Grade 1	4	4.5	5	6	7
S_2 = Grade 2	5	5.5	6	7	8
S_3 = Grade 3	6	6.5	7	7.5	8.5
S_4 = Grade 4	7	7.5	8	8.5	9
S_5 = Grade 5	8	8.5	9	10	11
S_6 = Grade 6	9	10	11	12	12
S_7 = Grade 7	10	11	12	13	13
S_8 = Grade 8	12	13	14	15	15

5. The basic allowances table (in thousands of dollars) table 9,

Table XIV

Skills / Grades	Skill 1	Skill 2	Skill 3	Skill 4	Skill 5	Skill 6
S_1 = Grade 1	.2	.1	0	1	0	0
S_2 = Grade 2	.4	.12	0	1.5	0	0
S_3 = Grade 3	.6	.15	0	2	.5	0
S_4 = Grade 4	.8	.2	.4	0	1.0	.2
S_5 = Grade 5	1	.3	.5	0	1.5	.6
S_6 = Grade 6	1.5	.5	.6	0	2	1
S_7 = Grade 7	2	.7	1	0	2.2	2
S_8 = Grade 8	3	1	2	0	2.4	4

and the above allowances increase by a rate of 10% of the basic allowance table annually to a maximum of 50% at the highest level.

REQUIRED

a. The organizational layers by paylevel during the fourth year
b. The salaries for each layer over the fourth year

c. The allowances over the same period
d. A calculation of the total budget.

Solution:

$$L^{(4)} = L_1\ L_2\ L_3\ L_4\ L_5$$

$$L_1 = \underline{H},\ L_2 = HQ,\ L_3 = HQ^2,\ L_4 = HQ^3,\ L_5 = S^{(0)}Q^4$$

Table XV

$L_1 = H =$

30	28	18	20	18	16	12	8
28	24	16	18	10	12	10	6
18	22	12	16	14	10	8	8
16	12	10	10	8	8	6	8
0	0	8	8	6	6	3	4
0	0	0	6	4	6	7	4

Table XVI

$L_2 = \underline{HQ} =$

22.5	27	19.1	18.6	16.6	13	7.6	3.8
21	23.6	16.8	16.7	10.6	9.4	6.2	2.9
13.5	20.1	13.4	14.4	13	8.4	5	3.6
12	12.2	9.9	9.5	7.6	6.4	3.8	3.5
0	0	6	7.6	5.8	4.8	2.1	1.75
0	0	0	4.5	4	4.6	4.1	1.95

Table XVII

$L_3 = HQ^2 =$

16.9	24.7	19.7	17.8	15.3	10.8	5.1	1.9
15.7	21.9	17.3	15.9	10.8	7.64	4.04	1.47
10.1	17.8	14.1	13.5	12	7.18	3.34	1.69
9	11.5	9.87	9.11	7.22	5.24	2.54	1.59
0	0	4.5	6.9	5.58	3.94	1.53	0.805
0	0	0	3.37	3.7	3.62	2.51	0.985

Table XVIII

$L_4 = HQ^3 =$

12.7	21.9	19.7	17.3	14.3	9.07	3.63	1.01
11.8	19.6	17.4	15.4	10.7	6.42	2.78	0.79
7.59	15.4	14.1	12.9	11.1	6.22	2.39	0.843
6.75	10.5	9.71	8.8	6.87	4.39	1.79	0.763
0	0	3.37	6.07	5.29	3.32	1.16	0.398
0	0	0	2.53	3.26	2.9	1.62	0.519

Table XVIX

$L_5 = S^0Q^4 =$							
12.7	23	21.9	19.7	15.8	8.86	3.16	0.714
9.49	18	17.2	15.5	13.5	8.01	3.27	0.737
6.33	14.7	14.3	12	10.8	7.37	3.06	0.697
6.33	9.91	9.24	9.95	9.98	6.13	2.29	0.499
0	0	6.33	12.4	11	5.39	1.66	0.315
0	0	0	3.8	5.58	3.76	1.22	0.205

Table XX
Salaries

Layer	1st Layer L_1C_1	2nd Layer L_2C_2	3rd Layer L_3C_3	4th Layer L_4C_4	5th Layer L_5C_5
Grade 1	368	310.5	258.75	232.88	243.63
Grade 2	430	455.95	455.85	471.32	524.81
Grade 3	384	423.8	458.36	482.29	586.12
Grade 4	546	534.75	532.12	535.35	659.93
Grade 5	480	489.6	491.22	515.09	732.61
Grade 6	522	466	422.18	387.89	474.24
Grade 7	460	316.8	228.72	173.78	190.49
Grade 8	456	227.5	118.16	64.935	47.507
Total	3646	3224.9	2965.4	2863.5	3459.3

Table XXI
Allowances

Level Grades	1st Layer	2nd Layer	3rd Layer	4th Layer	5th Layer
Grade 1	24.8	20.46	16.74	13.601	13.732
Grade 2	32.08	35.125	35.824	34.863	36.725
Grade 3	37.2	40.458	43.696	46.224	52.275
Grade 4	35.2	35.728	36.432	37.238	51.56
Grade 5	39.4	41.118	42.178	42.814	63.033
Grade 6	54	47.784	42.922	39.106	50.765
Grade 7	59.6	41.096	29.705	22.584	24.854
Grade 8	71.6	36.85	19.706	11.122	8.1885
Total	353.88	298.62	267.2	247.55	301.13

The required total budget is the sum of both tables XX and XXI shown below in Table XXII.

Table XXII

Grades	1st Layer	2nd Layer	3rd Layer	4th Layer	5th Layer	Total
Grade 1	392.8	330.96	275.49	246.47625	257.36484	1503.0911
Grade 2	462.08	491.0752	491.6736	506.18182	561.53756	2512.5482
Grade 3	421.2	464.258	502.0556	528.51127	638.39623	2554.4211
Grade 4	581.2	570.478	568.552	572.58684	711.48613	3004.303
Grade 5	519.4	530.718	533.3976	557.90381	795.63866	2937.0581
Grade 6	576	513.784	465.1016	426.99382	525.00522	2506.8846
Grade 7	519.6	357.896	258.4248	196.36838	215.34007	1547.6292
Grade 8	527.6	264.35	137.8664	76.05702	55.695752	1061.5692
Total	3999.88	3523.5192	3232.5616	3111.0792	3760.4645	17627.504

The data calculated for the previous example could be manipulated to yield more useful information at little cost. For example the layers L_1, L_2... L_5 could be summed along any of the three dimensions, skills, grades or level of salary to give information about the entire organization along the other two dimensions. Table XXIII below is a summation along the skill dimension to show the total inventory in terms of grades and level of salary.

Table XXIII

	1st Layer	2nd Layer	3rd Layer	4th Layer	5th Layer	Total
Grade 1	92	69	52	39	35	287
Grade 2	86	83	76	67	66	378
Grade 3	64	65	65	64	69	327
Grade 4	78	71	67	63	73	352
Grade 5	60	58	55	52	67	292
Grade 6	58	47	38	32	40	215
Grade 7	46	29	19	13	15	122
Grade 8	38	18	8	4	3	71
Total	522	440	380	334	368	2044

Dividing Table XXIII, the total inventory into Table XXII, the total budget, yields Table XXIV which is the average cost per person per year over all skills in a given grade and salary level during the subject time period.

Table XXIV

	1st Layer	2nd Layer	3rd Layer	4th Layer	5th Layer	Total
Grade 1	4.2696	4.7965	5.3235	6.3504	7.3945	5.2488
Grade 2	5.373	5.9237	6.4715	7.5178	8.5598	6.6503
Grade 3	6.5813	7.1205	7.6673	8.2188	9.2581	7.7893
Grade 4	7.4513	8.0011	8.5477	9.0912	9.7032	8.532
Grade 5	8.6567	9.2139	9.7728	10.831	11.946	10.118
Grade 6	9.931	11.025	12.118	13.21	13.285	11.669
Grade 7	11.296	12.427	13.558	14.689	14.696	12.698
Grade 8	13.884	15.106	16.335	17.569	17.585	14.86
Total	7.6626	8.0281	8.5027	9.2879	10.257	8.6296

Column 6 of Table XXIV is the result of dividing column 6 of Table XXIII into column 6 of Table XXII element by element, giving a weighted average of the annual cost per person per grade over all skills and all layers in that grade. Similarly, row 9 of Table XXIV indicates a weighted average of the annual cost per person per layer over all grades and all skills in that layer. The figure in row 9, column 9 of Table XXIV would indicate a weighted average of annual cost per person over all the grades, skills and layers.

These are a few examples of the possible sets of data the user might get by manipulating information obtained by the model; however, one must be cautioned that the model has an implicit assumption that everyone joins at the lowest level of salary in his grade, (i.e., no prior service is allowed for). This assumption simplifies the calculations because of the limit it imposes on the number of states, thus making $\underline{H}$, $\underline{S}$ and Q as small in size as possible. This assumption is also valid in most cases because inputs to the system from either the training facilities or promotions would join in the lowest level of their respective grades; however, individuals who join by transferring their services from other government agencies may bring some seniority in their grades with them. The model can accommodate this by adding more states to $\underline{S}$, $\underline{H}$ and Q, thus increasing the size of L along the grade dimension.

V. CONCLUSIONS

A. GENERAL

Markovian chains have been in use in various management related fields for many years. This thesis is a demonstration of their use in manpower planning. The discussion addressed the planning needs of a specific organization; however, the theme was kept general.

The model focused on two parts of the organization, namely the civilian grades and the training facilities, and the same logic and techniques are applicable to all other parts of the organization. One may visualize ten Ω matrices, representing the ten parts of the organization, or one large matrix representing the entire organization. For simplicity, the subject system, state 20 or state 18, was assumed to be isolated and the flows to either one were assumed to be directly controlled. This should be considered realistic because in steady state, the flows from the other parts of the organization to those states, as well as to other states, will eventually become constant. Those constant flows are variable through the manipulation of the model's elements.

The steady state explains how the organization might maintain a constant size and composition irrespective of time. This seems unrealistic because organizations in real life do grow; yet a planner may find it useful to visualize the expansion of the organization as a continuous movement from

one steady state to another. This explores different sets of policy changes that are necessary at different points in time if the organization's growth is to be kept under control.

B. THE MODEL ADVANTAGES

1. Simplicity

The model is simple enough, yet covers a wide range of situations. Because of the repetitive use of the same concepts, formula and calculations in each situation, the user rapidly builds up the required familiarity and expertise in handling the model.

The user does not have to be well versed in mathematics and probability theory to use the model. He will only have to know the parameters that are under his control and vary them through interaction with a computer terminal without worrying about the calculations.

The simplicity of the model combined with its interactive capability will tend to create a user's sensitivity to the long run implications of a policy change, thus serving an educational purpose.

2. Flexibility

The model demonstrates a Markovian chain application in each part of the organization; yet, whenever this concept proves invalid, the subject part can be isolated and modeled differently, and the interaction with the organization can resume within the general framework of the model.

The model demonstrates the extent of precision as a function of the Q matrix size. The size of the matrices

can change at the user's discretion depending on the nature of the problem being studied. For a more detailed solution, the user will have to use a more extensive partitioning scheme. This is due to the implicit assumptions that go with the Markovian chains.

C. LIMITATION

The limitations, in general terms, are represented by the wide gap between pure mathematics and real life situations. This section addresses two main reasons why the model may not simulate the actual reality and will touch on some remedies.

Both limitations stem from the implicit assumptions behind Markovian chains.

1. History of Previous Movements

Markovian chains assume that the movement to a state depends only on the state occupied during the previous period and is not affected at all by the prior history. This means that if $q_{6,7}$ is 0.3, for example, then 30% of those in state 6 will move to state 7 each period, regardless of how many are in either grade.

To get around this difficulty, the mathematical elements of the model need to be chosen as representative of the reality.

2. Influence of Human Behavior

Markovian chains have the implicit assumption that transition probabilities from a state are independent of the other transition probabilities. This may not be completely true in real life. To illustrate this point, suppose a

decision maker decided to decelerate promotions from one particular grade to the next higher. Then, according to the Markovian assumption, this would affect only the two elements in the Q matrix that represent the flow between the subject grade and its next higher. In reality other flows may be affected as well, for example an increase in attrition from the subject grade, the grade before or from the organization at large. This can compromise the model's result when large changes in promotion opportunities are considered. A planner must consider the human response to a policy change and try to represent all its dimensions by varying all the mathematical elements of the model that are likely to be affected.

D. SUMMARY

The model described in this paper, like all planning models, is best used for exploring trends in the organizational behavior over some time horizon.

It can be used for calculating actual stocks and flows especially over a short time horizon, provided that the model's elements were chosen as representative as possible of the real situation.

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